Anisentropic Gas Flow

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For one-dimensional plane, cylindrical, or spherical-symmetric anisentropic gas flow, relations between the rates of propagation of constant pressure, density, and temperature are established

1 Introduction

A DECAYING shock leaves behind an anisentropic gas flow, i e , one in which the specific entropy, though remaining constant for a particular fluid particle, varies from particle to particle The anisentropic flow of a perfect polytropic gas, devoid of viscosity and heat conductivity, and depending upon one space coordinate r, the distance measured from a fixed center, and upon time t, is governed by the following equations[1]:

$$\rho_t + \rho\, u + \rho u\ + (\sigma \rho u/r) = 0 \tag{1}$$

$$\rho(u_t + uu\,) + p\ = 0 \tag{2}$$

$$S_t + uS\ = 0 \tag{3}$$

$$p = \exp(s - s_0/c_p)\rho^\gamma \tag{4}$$

where the various symbols have their usual meaning The constant $\sigma = 0,1,2$ for the plane, cylindrical, or spherical-symmetric case Suffixes in the preceding equations and in sequel denote partial differentiation

2 Rates of Propagation of Constant Pressure and Density

Equation (3), in conjunction with (1) and (4), reduces to

$$\rho(p_t + up\,) + \gamma p[\rho u\ + (\sigma\rho u/r)] = 0 \tag{5}$$

In the r, t plane, the slope of isobars shall be given by

$$(dr/dt)_{p=\text{const}} = -(p_t/p_r)$$

which, with (2) and (5), yields

$$(dr/dt)_{p=\text{const}} = u + (\gamma p/p\,)[u\ + (\sigma u/r)]$$

If

$$R_p = (\gamma p/p\,)[u\ + (\sigma u/r)] \tag{6}$$

then R_p, so defined, represents the rate at which constant pressure is propagated with respect to the gas In a similar manner R_ρ, the rate at which constant density is propagated with respect to the gas, can be evaluated with the help of Eq (1) to be

$$R_\rho = (\rho/\rho\,)[u\ + (\sigma u/r)] \tag{7}$$

From (6) and (7) we have

$$R_p p\,/R_\rho \rho\ = \gamma p/\rho = c^2 \tag{8}$$

where c is the local speed of sound For an isentropic flow $p\ = c^2\rho$, and then from (8), $R_p = R_\rho$ as it should From (8) it follows that for a polytropic gas the local speed of sound is the geometric mean of the ratios R_p/R_ρ and $p\,/\rho$

3 Rate of Propagation of Absolute Temperature

For a perfect gas, $p = \rho RT$, where T is the absolute temperature and R the gas constant Accordingly, the slope of the isothermals in the r,t plane shall be given by

$$\left(\frac{dx}{dt}\right)_{T=\text{const}} = \left(\frac{\rho p_t - \rho_t p}{p\rho\ - \rho p}\right)$$

This, with (1) and (5), yields R_T, the rate at which absolute temperature is propagated with respect to the gas, as

$$R_T = \frac{(\gamma - 1)p\rho[u_r + (\sigma u/r)]}{(\rho p\ - p\rho\,)} \tag{9}$$

Equations (6–8) then give

$$(\gamma - 1)/R_T = \gamma/R_p - 1/R_\rho \tag{10}$$

a result first obtained by Ludford and Martin,[2] for the case $\sigma = 0$, from the formulation of plane ($\sigma = 0$) one-dimensional anisentropic gas flow as given by Martin [3] From (10), we have the result that for a plane, cylindrical, or spherical-symmetric anisentropic gas flow, the rate at which the absolute temperature is propagated with respect to the gas is a weighted harmonic mean of the rates at which density and pressure are propagated with respect to the gas

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Modification of Weierstrass-Erdmann Corner Conditions in Space Navigations

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Mayer's problem is of wide application in the optimization studies of space trajectories One of the two corner conditions due to Erdmann and Weierstrass gives the continuity of the partial derivatives of the Lagrange function with respect to such time derivatives of the space coordinates as are discontinuous at the corner If some of the space coordinates themselves are discontinuous there, then the partial derivatives of the Lagrange function, with respect to the time derivatives of the corresponding coordinates, must vanish on either side of the corner, in addition to being continuous This has been investigated and established in the following note A nontrivial application of this modified corner condition is under preparation and may be published soon

1 Introduction

MAYER'S problem in the calculus of variations provides an important tool for the determination of optimal trajectories of space vehicles An abstract theoretical formulation of the problem is given by Bliss [1] Lawden[2] has given a simplified formulation of the problem which is more suitable for the application to space dynamics Weierstrass-Erdmann corner conditions are derived from the concept of a corner as a point of discontinuity of the time derivatives of the space coordinates of the rocket The space coordinates themselves are treated as continuous at the corners The essential requirement of the continuity of a dynamical trajectory involves the idea of continuity of its cartesian space coordinates But if we use polar coordinates, or other forms of the generalized coordinates, we get discontinuities of space coordinates such as the vectorial angle θ without violating the essential requirement of the continuity of the trajectory

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itself This calls for a modification of Weierstrass-Erdmann corner conditions to take account of the fact that some of the space coordinates also may be discontinuous at the corners along with the discontinuity of the time derivatives of such coordinates This modification is the purpose of the present note

2 Mayer's Problem

Let a system of m ordinary differential equations of the first order be given by

$$\varphi_j(x_i,\dot{x}_i,t) = 0 \qquad (j = 1,2, \quad m;\ i = 1,2, \quad n;\ m < n) \tag{1}$$

with p equations of constraints

$$\psi_k(t_0,t_1,x_{i0},x_{i1}) = 0 \qquad (k = 1,2, \quad p \leq 2n + 2) \tag{2}$$

describing a certain mechanical system Here $x_i(t)(i = 1\ 2 \quad n)$ are the coordinates of the system These n unknown functions $x_i(t)$ satisfy Eqs (1) over a range of values of t in the interval $t_0 \leq t \leq t_1$, and the values taken by these functions at the terminal points of this interval are denoted by

$$x_i(t_0) = x_{i0} \qquad x_i(t_1) = x_{i1} \tag{3}$$

If $J = J(t_0,t_1,x_{i0},x_{i1})$, it is required to choose the functions x_i subject to the constraints (1) and (2) so that J is minimized Set

$$F = \lambda_j\varphi_j \tag{4}$$

with the usual summation convention, where $\lambda_j(t)$ are Lagrange multipliers Again let

$$I = J + \nu_k\psi_k + \int_{t_0}^{t_1} Fdt = H + \int_{t_1}^{t_0} Fdt \tag{5}$$

where $H = J + \nu_k\psi_k$ and ν_k's are constant Lagrange multipliers Thus, the problem of minimizing J reduces to the problem of minimizing I, since $F = \psi_k \equiv 0$ by Eqs (1) and (2)

3 Variation of I

In this section, we shall calculate the variation of the functional I for any virtual change in the path of integration between t_0 and t_1 with probable change of the terminal points t_0, t_1 themselves We assume the existence of a single corner t^* in $t_0 \leq t \leq t_1$, where some or all of the coordinates x_i and their time derivatives $\dot{x}_i$ may be discontinuous The probable discontinuity of some or all of the coordinates x_i is a new element in this note

The previous assumption splits the interval $t_0 \leq t \leq t_1$ into two subintervals, $t_0 \leq t < t^*$ and $t^* < t \leq t_1$, in which x_i and $\dot{x}_i$ are continuous We denote by $x_i^-(t)$, $\lambda_j^-(t)$ and $x_i^+(t)$, $\lambda_j^+(t)$ the values of these functions in the two subintervals, respectively

In the calculation of variation, it should be noted that at an ordinary point, i e , any point other than t_0, t_1, t^*, the variations are due to virtual changes in the coordinates x_i and $\dot{x}_i$ without any change in t itself These variations would be the same as if the integration were taken along a neighboring path rather than the actual one We shall indicate this variation by δ so that

$$\delta\varphi_j(x_i,\dot{x}_i,t) = \frac{\partial\varphi_j}{\partial x_i}\delta x_i + \frac{\partial\varphi_j}{\partial\dot{x}_i}\delta\dot{x}_i \tag{6}$$

But at the points t_0, t_1, and t^* there will be another kind of variation consisting of two parts, one part being due to the virtual variation of x_i and $\dot{x}_i$ and indicated by δ as in the forementioned, and the other part being due to a change in t and the consequent changes in x_i and $\dot{x}_i$ The total change will be denoted by Δ, so that

$$\Delta x_i(t) = \delta x_i(t) + \dot{x}_i(t)\delta t \tag{7}$$

Therefore,

$$\Delta I = \Delta H + \Delta\int_{t_0}^{t^*} F^-dt + \Delta\int_{t^*}^{t_1} F^+dt \tag{8}$$

Now

$$\Delta\int_{t_0}^{t^*} F^-dt = \delta\int_{t_0}^{t^*}\lambda_j^-\varphi_j^-dt + (\lambda_j^-\varphi_j^-)_{t^*}\Delta t^* - (\lambda_j^-\varphi_j^-)_{t_0}\Delta t_0 = \int_{t_0}^{t^*}\lambda_j^-\delta\varphi_j^-dt$$

Since the other parts of the variation vanish for $\varphi_j \equiv 0$ at t_0 and t^*, therefore (integrating the second integral by parts),

$$\begin{aligned}\Delta\int_{t_0}^{t^*} F^-dt &= \int_{t_0}^{t^*}\lambda_j^-\delta\varphi_j^-dt = \int_{t_0}^{t^*}\lambda_j^-\left[\frac{\partial\varphi_j}{\partial x_i^-}\delta x_i^- + \frac{\partial\varphi_j}{\partial\dot{x}_i^-}\delta\dot{x}_i^-\right]dt \\ &= \int_{t_0}^{t^*}\lambda_j^-\frac{\partial\varphi_j}{\partial x_i^-}\delta x_i^-dt + \lambda_j^-\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\delta x_i^-\Big|_{t_0}^{t^*} - \int_{t_0}^{t^*}\left[\dot{\lambda}_j^-\frac{\partial\varphi_j}{\partial\dot{x}_i^-} + \lambda_j^-\frac{d}{dt}\left(\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right)\right]\delta x_i^-dt \\ &= \int_{t_0}^{t^*}\left[[\lambda_j^-\frac{\partial\varphi_j}{\partial x_i^-} - \dot{\lambda}_j^-\frac{\partial\varphi_j}{\partial\dot{x}_i^-} - \lambda_j^-\frac{d}{dt}\left(\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right)\right]\delta x_i^-dt + \lambda_j^-(t^*)\times \\ &\quad \left[\frac{\partial\varphi_i}{\partial\dot{x}_i^-}\right]_{t^*}\delta x_i^-(t^*) - \lambda_j(t_0)\left[\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right]_{t_0}\delta x_{i0}\end{aligned}$$

Similarly for

$$\Delta\int_{t^*}^{t_1} F^+dt$$

and, using relation (7),

$$\begin{aligned}\Delta H &= \frac{\partial H}{\partial t_0}\delta t_0 + \frac{\partial H}{\partial x_{i0}}\Delta x_{i0} + \frac{\partial H}{\partial t_1}\delta t_1 + \frac{\partial H}{\partial x_{i1}}\Delta x_{i1} \\ &= \frac{\partial H}{\partial x_{i0}}\delta x_{i0} + \left(\frac{\partial H}{\partial t_0} + \frac{\partial H}{\partial x_{i0}}\quad\dot{x}_{i0}\right)\delta t_0 + \frac{\partial H}{\partial x_{i1}}\delta x_{i1} + \left(\frac{\partial H}{\partial t_1} + \frac{\partial H}{\partial x_{i1}}\quad\dot{x}_{i1}\right)\delta t_1\end{aligned}$$

Therefore,

$$\left.\begin{aligned}\Delta I &= \left(\frac{\partial H}{\partial t_0} + \frac{\partial H}{\partial x_{i0}}\quad\dot{x}_{i0}\right)\delta t_0 + \left(\frac{\partial H}{\partial t_1} + \frac{\partial H}{\partial x_{i1}}\quad\dot{x}_{i1}\right)\delta t_1 + \\ &\left(\frac{\partial H}{\partial x_{i0}} - \lambda_j(t_0)\left[\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right]_{t_0}\right)\delta x_{i0} + \left(\frac{\partial H}{\partial x_{i1}} + \lambda_j(t_1)\left[\frac{\partial\varphi_j}{\partial\dot{x}_i^+}\right]_{t_1}\right)\delta x_{i1} + \int_{t_0}^{t^*}\left[\lambda_j^-\frac{\partial\varphi_j}{\partial x_i^-} - \right. \\ &\left.\dot{\lambda}_j^-\frac{\partial\varphi_j}{\partial\dot{x}_i^-} - \lambda_j^-\frac{d}{dt}\left(\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right)\right]\delta x_i^-dt + \int_{t^*}^{t_1}\left[\lambda_j^+\frac{\partial\varphi_j}{\partial x_i^+} - \dot{\lambda}_j^+\frac{\partial\varphi_j}{\partial\dot{x}_i^+} - \right. \\ &\left.\lambda_j^+\frac{d}{dt}\left(\frac{\partial\varphi_j}{\partial\dot{x}_i^+}\right)\right]\delta x_i^+dt + \lambda_j^-(t^*)\times \\ &\left[\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right]_{t^*}\Delta x_i^-(t^*) - \lambda_j^+(t^*)\left[\frac{\partial\varphi_j}{\partial\dot{x}_i^+}\right]_{t^*}\Delta x_i^+(t^*) - \\ &\left[\lambda_j^-(t^*)\left(\frac{\partial\varphi_j}{\partial\dot{x}_i^-}\right)_{t^*}\dot{x}_i^-(t^*) - \lambda_j^+(t^*)\times\left(\frac{\partial\varphi_j}{\partial\dot{x}_i^+}\right)_{t^*}\dot{x}_i^+(t^*)\right]\delta t^*\end{aligned}\right\} \tag{9}$$

In this context, it is necessary to mention that corners were defined as a point of discontinuity of the time derivatives of the space coordinates, but the space coordinates themselves were supposed to be continuous there Thus, Troitskii[3] used the relations,

$$x_s^-(t^*) = x_s^+(t^*) \qquad \Delta x^-(t^*) = \Delta x^+(t^*)$$

But in this paper we deal with the different assumption that has been explained in the foregoing So, in view of the discontinuity of some or all of the space coordinates at the corner, we use the following relations:

$$x_s^-(t^*) \neq x_s^+(t^*) \qquad \Delta x_s^-(t^*) \neq \Delta x^+(t^*)$$

We note here that $\Delta x_i^\pm(t^*)$ are independent for $i = 1, \quad n$ and also that δt^*, $(2n - p)$ of the variations δx_{i0} and δx_{i1} and $2(n - m)$ of the variations $\delta x_i^\pm$ are independent Therefore, it is possible to select p ν_k's and $2m$ $\lambda_j^\pm$'s such that the coefficients of the remaining δx_{i0} and δx_{i1} and $\delta x_i^\pm$ in Eq (9) may be zero Thus, in Eq (9), the coefficients of all the variations δt_0, δt_1, δt^*, δx_{i0}, δx_{i1}, $\delta x_i^\pm$, $\Delta x_i^\pm(t^*)$ must vanish as though they were all independent Hence, we get the following equations from (9):

$$(\partial H/\partial t_0) + (\partial H/\partial x_{i0})\,\dot{x}_{i0} = 0 \qquad (10)$$

$$(\partial H/\partial t_1) + (\partial H/\partial x_{i1})\,\dot{x}_{i1} = 0 \qquad (11)$$

$$\frac{\partial H}{\partial x_{i0}} - \lambda_j(t_0)\left[\frac{\partial \varphi_j}{\partial \dot{x}_i}\right]_{t_0} = 0 \text{ or } \frac{\partial H}{\partial x_{i0}} - \left(\frac{\partial F}{\partial \dot{x}_i}\right)_0 = 0 \qquad (12)$$

$$\frac{\partial H}{\partial x_{i1}} + \lambda_j(t_1)\left[\frac{\partial \varphi_j}{\partial \dot{x}_i}\right]_{t_1} = 0 \text{ or } \frac{\partial H}{\partial x_{i1}} + \left(\frac{\partial F}{\partial \dot{x}_i}\right)_1 = 0 \qquad (13)$$

$$\lambda_j^\pm \frac{\partial \varphi_j}{\partial x_i^\pm} - \dot{\lambda}_j^\pm \frac{\partial \varphi_j}{\partial \dot{x}_i^\pm} - \lambda_j^\pm \frac{d}{dt}\left(\frac{\partial \varphi_j}{\partial \dot{x}_i^\pm}\right) = 0$$

$$\text{or} \quad \frac{\partial F}{\partial x_i^\pm} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i^\pm}\right) = 0 \qquad (14)$$

$$\left[\lambda_j^- \frac{\partial \varphi j}{\partial \dot{x}_i^-}\dot{x}_i^- - \lambda_j^+ \frac{\partial \varphi_j}{\partial \dot{x}_i^+}\dot{x}_i^+\right]_{t^*} = 0$$

$$\text{or} \quad \left[\dot{x}_i^- \frac{\partial F}{\partial \dot{x}_i^-} - \dot{x}_i^+ \frac{\partial F}{\partial \dot{x}_i^+}\right]_{t^*} = 0 \qquad (15)$$

$$\left[\lambda_j^\pm \frac{\partial \varphi_j}{\partial \dot{x}_i^\pm}\right]_{t^*} = 0 \quad \text{or} \quad \left[\frac{\partial F}{\partial \dot{x}_i^\pm}\right]_{t^*} = 0 \qquad (16)$$

Equations (10–14) are the governing equations of the system and are the same here as in Ref 2 Equations (15) and (16) are the corner conditions The condition in (15) means that $\dot{x}_i(\partial F/\partial \dot{x}_i)$ is continuous across the corner This is identical with the corresponding condition as stated in Ref 2, namely, that $F - \dot{x}_i(\partial F/\partial \dot{x}_i)$ is continuous at the corner ($F \equiv 0$ everywhere)

Equation (16) means that $\partial F/\partial \dot{x}_i$ is continuous through the value zero at the corner The classical corner condition as stated in Ref 2 gives only the continuity of $\partial F/\partial \dot{x}_i$ at the corner, but it does not give the vanishing of this partial derivative on either side of the corner This modification of the corner condition is due to the discontinuity of x_i at the corner and is available only for such space coordinates x_i as are discontinuous there For other space coordinates, only the classical condition of continuity is available, provided their time deiivatives are discontinuous A nontrivial application of this modified corner condition is under preparation and may be published soon

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Additional Modes of Instability for Poiseuille Flow over Flexible Walls

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THE modification of Tollmein-Schlichting waves by flexible walls was presented by Hains and Price[1] for two-dimensional Poiseuille flow The possibility of the appearance of other modes of instability was examined by Brooke-Benjamin [2] Using the asymptotic theory, Landahl[3] computed stability diagrams for the Blasius profile This note will compare the stability curves of Refs 1 and 3 and present additional eigenvalues for Poiseuille flow which lead to instability

The stability diagram (Fig 1) shows the modification of Tollmein-Schlichting waves with wall flexibility The stability curves close and the region of instabil ty becomes progressively smaller as the flexibility increases until complete stability results In agreement with the trends given in Ref 2, Landahl[3] found that wall flexibility displaced the stability curve for Tollmein-Schlichting waves to lower wave numbers α and slightly higher Reynolds number R No mention is made of the closing and shrinking of the stability curves

These divergent results are probably due to how the wall properties are nondimensionalized The no-slip condition at the flexible surface leads to the relation

$$\xi(d^3\phi/dy^3) + \phi = 0 \qquad (1)$$

which reduces to the condition for a rigid wall when $\xi = 0$ Depending on the free parameter chosen in R, ξ is given by one of the four relations in Table 1 The wall is a membrane stretched with tension T and has a damping coefficient D The nondimensional forms of these quantities are K_3 and K_2, respectively, given in Table 1 The fluid density is ρ, and the viscosity is μ For Poiseuille flow, U_∞ is the maximum velocity, and L is the channel half-width For the Blasius profile, U_∞ is the freestream velocity, and L is the boundary-layer thickness

Hains and Price chose U_∞ as the free parameter, whereas Landahl chose μ Apparently, the latter choice was made in order to obtain a boundary condition that is independent of R The exact boundary condition given by Eq (1) is independent of R if ρ is chosen as the free parameter An approximation to Eq (1) was used in Ref 3 by employing the inviscid equa-

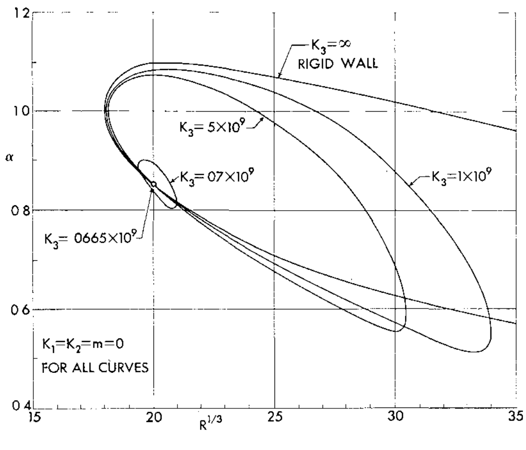

Fig 1 Stability diagıams for the Tollmein-Schlichting mode foı Poiseuille flow with walls with increased flexibility Wall damping, mass, and elastic foundation neglected *R* changed by variation of U_∞ only (from Ref 1)

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